Optics Letters

Universal self-similar asymptotic behavior of optical bump spreading in random medium atop incoherent background

Xiaofei Li,[1,2] Sergey A. Ponomarenko,[3,4] Zhiheng Xu,[1,2] Fei Wang,[5] Yangjian Cai,[1,2,5,6] and Chunhao Liang[1,2,3,7]

[1]Shandong Provincial Engineering and Technical Center of Light Manipulation & Shandong Provincial Key Laboratory of Optics and Photonic Devices, School of Physics and Electronics, Shandong Normal University, Jinan 250014, China
[2]Collaborative Innovation Center of Light Manipulations and Applications, Shandong Normal University, Jinan 250358, China
[3]Department of Electrical and Computer Engineering, Dalhousie University, Halifax, Nova Scotia, B3J 2X4, Canada
[4]Department of Physics and Atmospheric Science, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada
[5]School of Physical Science and Technology, Soochow University, Suzhou 215006, China
[6]e-mail: yangjiancai@suda.edu.cn
[7]e-mail: cliang@dal.ca

Received 6 December 2019; revised 21 December 2019; accepted 21 December 2019; posted 23 December 2019 (Doc. ID 385246); published 30 January 2020

We demonstrate theoretically that the average spatial intensity profile of any partially coherent optical beam, composed of a finite-power bright intensity bump atop a fluctuating background, evolves into a universal self-similar Gaussian shape upon long-term propagation in a statistically homogeneous, isotropic linear random medium. The result depends neither on the degree of the background spatial coherence nor on the strength of the medium turbulence. To our knowledge, this is the first demonstration of universal self-similar asymptotics in linear random media.

https://doi.org/10.1364/OL.385246

Introduction. Studies of laser beam propagation through linear random media have received fresh impetus owning to the advancement of modern day free-space optical communication protocols that involve laser beams propagating through atmospheric turbulence [1–3]. To date, there exists an extensive body of research on propagation of coherent and partially coherent beams through atmospheric turbulence; for a recent review, see, e.g., [4]. Despite impressive progress though, there are but a handful of publications [5–7] demonstrating any generic characteristics of laser beams, independent of their initial spatial profiles or source states of coherence, on beam propagation in random media. In particular, the authors of Ref. [5] demonstrated that the root mean square (rms) width of any beam generated by a spatially coherent source obeys a universal propagation law in a statistically homogenous, isotropic random medium, with the long-term rms evolution being determined solely by the medium turbulence. The result of [5] was later extended to the beams produced by partially coherent sources [6,7].

Yet, the universality properties discussed in Refs. [6,7] are limited to statistically homogeneous laser beams, such that their degree of coherence at a pair of space points depends only on the distance between the points. At the same time, several classes of non-uniformly correlated statistical beams have been theoretically proposed [8–11] and experimentally realized [11–14]. These beams have attracted growing attention for their unusual robustness against the adverse medium turbulence effects [15]. In particular, dark and antidark diffraction free beams, theoretically discovered in Ref. [9] and generated in the lab in Ref. [14], have lately piqued researchers' curiosity due to their ability to nearly defy free-space diffraction over distances that can be easily controlled [14,16,17]. These beams manifest themselves as dips or bumps against statistically homogeneous backgrounds. Several generalizations of such beams were proposed [16,17], including, notably, white-light (polychromatic) dark/antidark fields residing on an incoherent background [16]. These developments trigger a natural question: can bumps/dips atop an incoherent background exhibit universal propagation scenarios in linear random media?

In this Letter, we show that the average spatial profile of any finite-power random intensity bump on a statistically homogeneous background asymptotically approaches a self-similar Gaussian shape as the composite beam propagates in a statistically homogeneous, isotropic linear random medium. The rms width of the universal self-similar asymptotics is determined entirely by the medium turbulence. We stress that our result is independent of either the optical source particulars or the strength of the medium turbulence. We also emphasize that the discovered universal asymptotic scenario comes about at distances so long that the initial conditions at the source play no role, yet so short that the paraxial approximation is still valid. Thus, the discovered regime can be classified as *self-similar intermediate asymptotics* in the sense elucidated in Ref. [18].

Convolution representation in turbulent media. Let us start by recalling some key properties of dark or antidark diffraction-free beams. We demonstrated in Ref. [9] that the cross-spectral density of any partially coherent diffraction-free beam in free space is necessarily of the form

$$W(\mathbf{r}_1, \mathbf{r}_2) = \Phi(\mathbf{r}_-) + \Psi(\mathbf{r}_+), \tag{1}$$

where $\mathbf{r}$ stands for a radius vector in the plane transverse to the beam axis z, and we introduced the difference and average position variables as

$$\mathbf{r}_- = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{r}_+ = (\mathbf{r}_1 + \mathbf{r}_2)/2. \tag{2}$$

Notice that the average intensity reads

$$I(\mathbf{r}) \equiv W(\mathbf{r}, \mathbf{r}) = \Phi(0) + \Psi(\mathbf{r}), \tag{3}$$

where $\Psi(\mathbf{r})$ represents the spatial profile of a "bump/dip" against a constant "background" $\Phi(0)$.

As a consequence of the cross-spectral density Hermiticity, the arbitrary functions Φ and Ψ obey the constraints

$$\Psi^*(\mathbf{r}_+) = \Psi(\mathbf{r}_+), \quad \Phi^*(-\mathbf{r}_-) = \Phi(\mathbf{r}_-). \tag{4}$$

In addition, the cross-spectral density must be non-negative definite, which is a nontrivial constraint, in general. One way to satisfy it is to find a series expansion of W in terms of coherent modes with nonnegative coefficients (modal weights). Unfortunately, this can be accomplished analytically in only a few cases [8,9]. Alternatively, we can express W in terms of the angular spectra as

$$W(\mathbf{r}_-, \mathbf{r}_+) = \int \mathrm{d}^2\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}_-} \mathcal{A}_\Phi(\mathbf{k}) + \int \mathrm{d}^2\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}_+} \mathcal{A}_\Psi(\mathbf{q}), \tag{5}$$

and in the case of bumps atop any background, the nonnegative definiteness of W is guaranteed by the virtue of Bochner's theorem [19] for real nonnegative angular spectra, such that

$$\mathcal{A}_\Phi(\mathbf{k}) \geq 0; \quad \mathcal{A}_\Psi(\mathbf{q}) \geq 0. \tag{6}$$

Hereafter, we will focus on partially coherent bumps riding on a fluctuating background.

Under the usual conditions of negligible backscattering, the average intensity distribution of any beam in any transverse plane, $z \geq 0$ in a statistically homogeneous, isotropic random medium is given by [1]

$$I(\mathbf{r}, z) = \left(\frac{k}{2\pi z}\right)^2 \int \mathrm{d}^2\mathbf{r}_1 \int \mathrm{d}^2\mathbf{r}_2 W^{(0)}(\mathbf{r}_1, \mathbf{r}_2) \times \exp\left[-\frac{ik}{2z}(\mathbf{r}_1^2 - \mathbf{r}_2^2) - \frac{ik}{z}(\mathbf{r}_2 - \mathbf{r}_1)\cdot\mathbf{r}\right]\Gamma_m(|\mathbf{r}_1 - \mathbf{r}_2|, z). \tag{7}$$

Here, Γ_m is a two-point correlation function of the random phases introduced into each field realization due to the medium fluctuations. For statistically homogeneous, isotropic medium fluctuations, we have, under the most general conditions [1],

$$\Gamma_m(|\mathbf{r}_1 - \mathbf{r}_2|, z) = \exp\left\{-4\pi^2 k^2 z \int_0^1 \mathrm{d}\xi \int_0^\infty \mathrm{d}\kappa \kappa S_n(\kappa) \times [1 - J_0[\kappa(1-\xi)|\mathbf{r}_1 - \mathbf{r}_2|)]]\right\}, \tag{8}$$

where $S_n(\kappa)$ is a spatial spectrum of the medium fluctuations, and $J_0(x)$ is a zero-order Bessel function of the first kind.

Assuming a partially coherent bump on a background at the source,

$$W^{(0)}(\mathbf{r}_1, \mathbf{r}_2) = W^{(0)}(\mathbf{r}_-, \mathbf{r}_+) = \Phi(\mathbf{r}_-) + \Psi(\mathbf{r}_+), \tag{9}$$

and transforming to the difference and average position variables given by Eq. (2) in Eq. (7), we arrive at

$$I(\mathbf{r}, z) = \left(\frac{k}{2\pi z}\right)^2 \int \mathrm{d}^2\mathbf{r}_- \exp[ik(\mathbf{r}_-\cdot\mathbf{r})/z]\Gamma_m(\mathbf{r}_-, z) \times \int \mathrm{d}^2\mathbf{r}_+ e^{ik(\mathbf{r}_-\cdot\mathbf{r}_+)/z} W^{(0)}(\mathbf{r}_-, \mathbf{r}_+). \tag{10}$$

On substituting from Eq. (9) into Eq. (10), expressing the medium correlation function via its Fourier transform through

$$\Gamma_m(|\mathbf{r}|, z) = \left(\frac{k}{z}\right)^2 \int \mathrm{d}^2\mathbf{r}' \exp[ik(\mathbf{r}'\cdot\mathbf{r})/z]\tilde{\Gamma}_m\left(k|\mathbf{r}'|/z, z\right), \tag{11}$$

and using the integral representation of a 2D delta function as

$$\delta(k\mathbf{r}/z) = \int \frac{\mathrm{d}^2\mathbf{R}}{(2\pi)^2} \exp[ik(\mathbf{r}\cdot\mathbf{R})/z], \tag{12}$$

we arrive, after elementary algebra, at the (arguably) elegant and physically intuitive expression for the average intensity distribution of a propagated beam in the form

$$I(\mathbf{r}, z) = I_0 + \frac{k^2}{z^2} \int \mathrm{d}^2\mathbf{r}' \Psi(\mathbf{r} - \mathbf{r}')\tilde{\Gamma}_m\left(k|\mathbf{r}'|/z, z\right), \tag{13}$$

where $I_0 = \Phi(0)$ is the average background intensity, and the tilde denotes a Fourier transform in Eqs. (11) and (13).

Universal self-similar asymptotics. Equation (13) reveals the physics of any random bump propagation through turbulence. First, we notice that the first (background) term remains unaffected by the turbulence, as is expected, because it is essentially an incoherent plane wave. The second (bump) term is a convolution of the initial bump profile and a scaled correlation function of the medium. Notice that, as is readily inferred from Eq. (13), the composite beam intensity is determined entirely by the medium turbulence, which is a direct consequence of the source diffraction-free nature in the absence of the medium fluctuations.

A brief inspection of the medium correlation function, Eq. (8), reveals that at short propagation distances, it is very flat, implying that its Fourier transform is nearly a delta function. Indeed, as $z \to 0$, $\Gamma_m(|\mathbf{r}|, z) \to 1$. Thus, $\lim_{z\to 0} (k/z)^2 \tilde{\Gamma}_m(k\mathbf{r}/z, z) = (k^2/z^2)\delta(k\mathbf{r}/z) = \delta(\mathbf{r})$, yielding a source bump profile in Eq. (13). On the other hand, at long propagation distances, the medium correlation function becomes very narrow, implying an extremely broad Fourier transform. Hence, if we introduce the total power carried by the bump,

$$\Delta P = \int \mathrm{d}^2\mathbf{r}\Psi(\mathbf{r}), \tag{14}$$

we conclude that at a long propagation distance, a bump of any profile can be approximated as $\Psi(\mathbf{r} - \mathbf{r}') \simeq \Delta P \delta(\mathbf{r} - \mathbf{r}')$, yielding the asymptotic average intensity profile as

$$I_\infty(\mathbf{r}, z) \simeq I_0 + \frac{k^2 \Delta P}{z^2} \tilde{\Gamma}_m \left(\frac{k|\mathbf{r}|}{z}, z \right). \quad (15)$$

Next, at sufficiently long propagation distances, only a small spatial region around the beam axis contributes to Γ_m, implying that the Bessel function in Eq. (8) can be expanded into a Taylor series. Keeping only the leading orders, $J_0(x) \simeq 1 - x^2/4$, we may approximate the correlation function by a Gaussian as

$$\Gamma_m(|\mathbf{r}|, z) = \exp\left[-\frac{|\mathbf{r}|^2}{2\sigma_c^2(z)} \right], \quad (16)$$

where the medium correlation length $\sigma_c(z)$ is defined by the expression

$$\sigma_c^2(z) = \frac{3}{2\pi^2 k^2 z \int_0^\infty d\kappa \kappa^3 S_n(\kappa)}. \quad (17)$$

On taking a Fourier transform in Eq. (16), using Eq. (17), and substituting into Eq. (15), we finally arrive at

$$I_\infty(\mathbf{r}, z) = I_0 + \frac{\Delta P}{\pi \sigma_\infty^2(z)} \exp\left[-\frac{|\mathbf{r}|^2}{\sigma_\infty^2(z)} \right]. \quad (18)$$

Here, we introduce an rms width of the bump asymptotics by the expression

$$\sigma_\infty^2(z) = \frac{4\pi^2}{3} z^3 \int_0^\infty d\kappa \kappa^3 S_n(\kappa). \quad (19)$$

Equations (18) and (19) furnish a *universal self-similar asymptotics* of a finite-power bump of any initial profile and degree of spatial coherence, propagating atop an incoherent background in a statistically homogeneous, isotropic linear random medium. This is the key result of this Letter.

Two instructive conclusions can be drawn from Eqs. (18) and (19). First, the rms width of the universal asymptotic profile depends only on the turbulence strength, as it should. Second, Eq. (19) gives $\sqrt{2}$ times the rms width derived in Ref. [6] for statistically uniform beams. It follows that although the studied random bumps defy diffraction in free space, they eventually succumb to the medium turbulence and spread somewhat faster than their statistically uniform counterparts in the long term. This apparently represents a flip side of using the source that is diffraction free in the absence of turbulence for optical communications through turbulence. Yet, the statistical bumps on an incoherent background appear to be remarkably structurally resilient to turbulence by yielding the universal self-similar asymptotic profile over sufficiently long propagation distances. We are unaware of any analogous results for conventional statistically uniform beams.

The random bump evolution toward the universal self-similar profile can be examined by evaluating its rms width [3]. Incidentally, the evolution of the latter with z determines a characteristic range of long enough distances over which our asymptotics works for a given input beam profile. Assuming, without loss of generality, the bump to be symmetric with respect to the propagation axis, its rms width $\sigma_{bp}(z)$ is defined as

$$\sigma_{bp}^2(z) = \frac{\int d^2\mathbf{r} \mathbf{r}^2 [I(\mathbf{r}, z) - I_0]}{\int d^2\mathbf{r} [I(\mathbf{r}, z) - I_0]}. \quad (20)$$

By energy conservation, the denominator is just the total power carried by the bump:

$$\int d^2\mathbf{r} [I(\mathbf{r}, z) - I_0] = \int d^2\mathbf{r} \Psi(\mathbf{r}) = \Delta P. \quad (21)$$

This can be formally obtained from Eq. (13) by interchanging the integration order, shifting the integration variable, and noticing that $(k^2/z^2) \int d^2\mathbf{r}' \tilde{\Gamma}_m(k|\mathbf{r}'|/z, z) = \Gamma_m(0, z) = 1$. Further, assuming the source profile is axially symmetric, $\Psi(\mathbf{r}) = \Psi(r)$, we obtain

$$\sigma_{bp}^2(z) = \left(\frac{2\pi}{\Delta P} \right) \int_0^\infty dr r^3 [I(r, z) - I_0]. \quad (22)$$

Next, we evaluate numerically the rms width evolution of the beams generated by a couple of random bump sources. In our numerical calculations, we assume the medium to be the turbulent atmosphere that is most relevant to free-space optical communications. We apply the Kolmogorov–von Karman model for the medium turbulence spectrum [1]:

$$S_n(\kappa) = 0.033 C_n^2 \frac{\exp(-\kappa^2/\kappa_m^2)}{(\kappa^2 + \kappa_0^2)^{11/6}}, \quad (23)$$

where C_n^2 is a so-called structure constant, $\kappa_m = 5.92/l_0$, and $\kappa_0 = 4\pi/L_0$; l_0 and L_0 are the inner and outer scales of the turbulence, respectively. We take $C_n^2 = 0.5 \times 10^{-13}\ \text{m}^{-2/3}$, $l_0 = 0.1$ m, and $L_0 = 1$ m corresponding to the strong fluctuation regime of the atmosphere [1,20].

Let us first consider a Gaussian bump of the initial width σ_0,

$$\Psi(\mathbf{r}) = \frac{\Delta P}{\pi \sigma_0^2} \exp\left(-\frac{\mathbf{r}^2}{\sigma_0^2} \right). \quad (24)$$

The rms width behavior of the bump is illustrated in Fig. 1 in a red curve, whereas the rms width of the universal self-similar profile is shown in blue dots. We can infer at once from the figure that the rms width of the Gaussian bump monotonously attains the asymptotic spreading law of Eq. (19). This is not surprising, as an initially Gaussian bump is expected to maintain—more or less—its Gaussian profile on propagation through turbulence.

Let us now consider an initial bump profile with a long tail,

$$\Psi(\mathbf{r}) = \frac{\Delta P}{2\pi} \frac{\sigma_0}{(\sigma_0^2 + r^2)^{3/2}}, \quad (25)$$

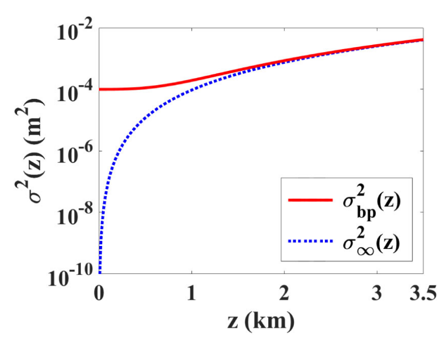

Fig. 1. Rms width of a Gaussian bump propagating in atmospheric turbulence as a function of propagation distance (solid red curve), and rms width of the universal self-similar profile (blue dots). Rms width at the source is 0.01 m. The parameters of the bump and the turbulence are: $\sigma_0 = 0.01$ m, $\lambda = 532$ nm, $C_n^2 = 0.5 \times 10^{-13}\ \text{m}^{-2/3}$, $l_0 = 0.1$ m, and $L_0 = 1$ m.

which has a manifestly non-Gaussian shape at the source. This is a bona fide (nonnegative definite) bump because, as can be easily verified, its profile results from a Fourier transform of the real nonnegative angular spectrum:

$$\mathcal{A}(\mathbf{q}) = \frac{\Delta P}{2\pi} \exp(-\sigma_0|\mathbf{q}|). \tag{26}$$

Notice that the initial rms width of such a bump is infinite: the bump has a very long tail indeed. We exhibit the rms evolution of the long-tail bump and that of the universal self-similar profile in Fig. 2 in the solid red and dotted blue curves, respectively. It can be inferred from the figure that the bump rms width will initially decrease, attaining a minimum, followed by an asymptotical approach toward $\sigma_\infty(z)$. The region around the minimum width is displayed in the inset to the figure in red dots. This rms width behavior implies dramatic reshaping of the bump spatial profile. To verify that the bump shape, not just its rms width, approaches its self-similar asymptotics, we

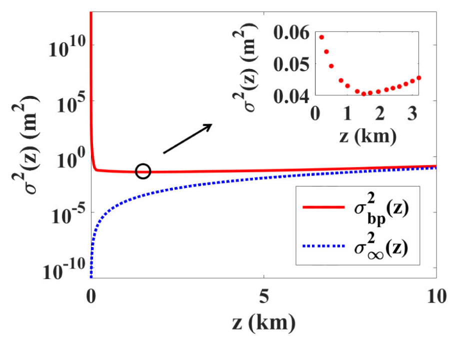

Fig. 2. Rms width of a Lorentzian bump propagating in atmospheric turbulence as a function of propagation distance (solid red curve), and rms width of the universal self-similar profile (blue dots). All parameters are the same as those in Fig. 1.

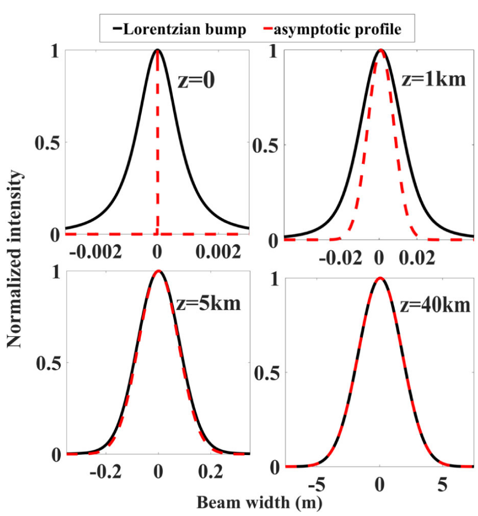

Fig. 3. Evolution of the normalized average intensity of a Lorentzian bump (solid black curve) versus the self-similar asymptotic intensity (dashed red curve) on propagation in atmospheric turbulence. All parameters are the same as those in Fig. 1.

show the evolution of the normalized intensity profile of the bump in Fig. 3 at several propagation distances and compare it with the universal self-similar asymptotic profile. It is seen in the figure that the two profiles virtually coincide at the propagation distance of 40 km for realistic atmospheric turbulence conditions.

Conclusion. We examined the average intensity profile evolution of random bumps on an incoherent background in statistically homogeneous, isotropic linear random media. We have shown that any such bump, carrying a finite power, evolves asymptotically into a universal self-similar Gaussian. Our results are independent of the spatial shape and coherence state of the bump at the source, and they do not depend on the strength of the medium turbulence. We expect our results to trigger further interest in universal asymptotic properties of statistical beams in random media as well as to inform any follow-up research on optical communications through turbulent atmosphere.

Funding. Natural Sciences and Engineering Research Council of Canada (RGPIN-2018-05497); National Natural Science Foundation of China (11525418, 91750201, 11874046, 11974218, 11904247, 11947239); Innovation Group of Jinan (2018GXRC010); China Postdoctoral Science Foundation (2019M662424).

Disclosures. The authors declare no conflicts of interest.

REFERENCES

1. L. C. Andrews and R. L. Phillips, *Laser Beam Propagation through Random Media* (SPIE, 1998).
2. F. Wang, X. Liu, and Y. Cai, Prog. Electromagn. Res. **150**, 123 (2015).
3. Z. Zhang, X. Liang, M. Goutsoulas, D. Li, X. Yang, S. Yin, J. Xu, D. N. Christodoulides, N. K. Efremidis, and Z. Chen, APL. Photon. **4**, 076103 (2019).
4. G. Gbur, J. Opt. Soc. Am. A **31**, 2038 (2014).
5. Z. I. Feizulin and Yu. A. Kravtsov, Radiophys. Quantum Electronics **10**, 33 (1967).
6. G. Gbur and E. Wolf, J. Opt. Soc. Am. A **19**, 1592 (2002).
7. S. A. Ponomarenko, J. J. Greffet, and E. Wolf, Opt. Commun. **208**, 1 (2002).
8. S. A. Ponomarenko, J. Opt. Soc. Am. A **18**, 150 (2001).
9. S. A. Ponomarenko, W. Huang, and M. Cada, Opt. Lett. **32**, 2508 (2007).
10. H. Lajunen and T. Saastamoinen, Opt. Lett. **36**, 4104 (2011).
11. A. S. Ostrovsky, J. Garcia-Garcia, C. Rickenstorff-Parrao, and M. A. Olvera-Santamaria, Opt. Lett. **42**, 5182 (2017).
12. G. V. Bogatyryova, C. V. Felde, P. V. Polyanskii, S. A. Ponomarenko, M. S. Soskin, and E. Wolf, Opt. Lett. **28**, 878 (2003).
13. X. Chen, J. Li, S. M. H. Rafsanjani, and O. Korotkova, Opt. Lett. **43**, 3590 (2018).
14. X. Zhu, F. Wang, C. Zhao, Y. Cai, and S. A. Ponomarenko, Opt. Lett. **44**, 2260 (2019).
15. J. Li, X. Chen, S. McDuffie, A. M. Najar, S. M. H. Rafsanjani, and O. Korotkova, Opt. Commun. **446**, 178 (2019).
16. M. Yessenov, B. Bhaduri, H. E. Kondakci, M. Meem, R. Menon, and A. F. Abouraddy, Optica **6**, 598 (2019).
17. M. W. Hyde and S. Avramov-Zamourovich, J. Opt. Soc. Am. A **36**, 1058 (2019).
18. G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge University, 1996).
19. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, 1995).
20. F. Wang and O. Korotkova, Opt. Lett. **41**, 1546 (2016).